Temporal Recursion in the UNNS Substrate and Its Klein Surface Realization

UNNS Working Note

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Abstract

We refine the notion of *temporal recursion* in the UNNS substrate (Unbounded Nested Number Sequences) by relating the global geometry of time-depth evolution to a non-orientable quotient: the Klein surface (Klein bottle). Locally, forward recursion is governed by a time-step map F, while (partial) reverse recursion is encoded by local inverses F^{-1} on stability domains. Globally, the presence of a *time-reversal involution* S that conjugates F to F^{-1} and a periodic stroboscopic section produces an identification of a time–phase cylinder with a *glide reflection*, yielding a Klein surface when quotiented. This formalizes when temporal backtracking is locally consistent but globally obstructed by non-orientability (captured by $w_1 \neq 0$). We give diagnostic criteria in terms of Floquet monodromy, parity of orientation, and UNNS operator symmetries, and we provide two TikZ diagrams: (i) forward/reverse recursion cones and (ii) the Klein identification rectangle.

1 Preliminaries: Temporal Recursion in UNNS

Let $\mathcal{X}$ be a state space (finite- or infinite-dimensional, discrete or continuous), and let a UNNS evolution be given by

$$x_{n+1} \;=\; F(x_n; \Theta), \qquad n \in \mathbb{Z}, \tag{1}$$

where Θ collects UNNS operators (e.g. damping α, drift δ, collapse threshold ε, inlaying lattice scale h, etc.). We interpret the *depth* $n \in \mathbb{N}$ as the UNNS notion of time.

Definition 1 (Local reversibility domain)**.** *A subset $U \subset \mathcal{X}$ is a* reversibility domain *if there exists a map $F^{-1} : F(U) \to U$ such that $F^{-1}(F(x)) = x$ for all $x \in U$. We say temporal recursion is* locally invertible *on U.*

Remark 1 (Global obstructions)**.** *Global invertibility may fail because of (i) non-injective F (many-to-one collapse), (ii) absorbing sets (e.g. ε-collapse to 0), or (iii) topological obstructions introduced by symmetry operations that reverse orientation in a periodic stroboscopic section, as developed below.*

2 Stroboscopic Sections, Symmetries, and Time Reversal

Assume there is a periodic section of depth $T \in \mathbb{N}$ and an internal phase variable $\theta \in S^1$ (e.g. a normalized iteration phase, angle on a Poincaré section, or a UNNS echo-phase) such that the pair (n, θ) coordinatizes a *time–phase cylinder* $C = S^1_\theta \times \mathbb{Z}_n$ modulo period T:

$$(n, \theta) \sim (n + T, \theta).$$

Suppose further there is an involution $S : \mathcal{X} \to \mathcal{X}$ with $S^2 = \mathrm{id}$ that implements a *time-reversal* symmetry in the sense

$$S \circ F \circ S \;=\; F^{-1} \quad \text{on a reversibility domain.} \tag{2}$$

This captures the idea that applying S "flips" the local temporal arrow.

Definition 2 (Orientation parity of the monodromy)**.** *Let M denote the Floquet (depth-T) monodromy on a tangent (or linearized) space along a periodic UNNS orbit. We say the stroboscopic section is* orientation preserving *if* $\det M > 0$ *and* orientation reversing *if* $\det M < 0$.

3 From Cylinder to Klein: The Gluing That Obstructs Global Reversal

The classical Klein bottle K arises from the rectangle $[0,1] \times [0,1]$ with identifications

$$(x,0) \sim (x,1), \qquad (0,y) \sim (1,1-y).$$

Equivalently, it is a cylinder with a *glide reflection* on the second identification.

In our UNNS time–phase cylinder, the depth-T identification $(n,\theta) \sim (n+T,\theta)$ always holds. If, in addition, the time-reversal symmetry (2) acts as

$$(n,\theta) \sim (n,\ 1-\theta) \quad \text{upon traversing the } S\text{-edge}, \tag{3}$$

then the composite quotient produces a non-orientable surface. When the first identification is purely periodic in n and the second identification flips $\theta \mapsto 1-\theta$, the quotient is (topologically) a Klein bottle.

Proposition 1 (Klein regime)**.** *Assume:*

1. *There exists a stroboscopic period T (depth-T return).*
2. *The time-reversal map S satisfies* (2) *on a reversibility domain that intersects the stroboscopic orbit.*
3. *The induced action on the phase coordinate is an involution $\theta \mapsto 1-\theta$ (orientation reversal in phase).*

Then the global time–phase quotient of the UNNS evolution under $\{(n,\theta) \sim (n+T,\theta),\ (n,0) \sim (n,1),\ (0,\theta) \sim (T,1-\theta)\}$ is a Klein surface. In particular, the first Stiefel–Whitney class w_1 is nonzero, and the time bundle is non-orientable.

Remark 2 (Interpretation)**.** *Locally, a reverse step F^{-1} exists (on the domain of reversibility), but globally any attempt to patch a single-valued time orientation across the quotient fails. Thus,* local time travel *exists (reversible steps), yet* global time orientation *is obstructed by non-orientability. The obstruction is topological (captured by $w_1 \neq 0$), not merely metric.*

4 A Criterion via Floquet Monodromy and UNNS Operators

Let DF denote the linearization of F along a periodic UNNS orbit and $M := DF^T$ its T-step product.

Theorem 1 (Orientation diagnostic)**.** *If* $\det M < 0$ *(orientation reversing monodromy) and the UNNS operator set Θ admits an involution S with $S^2 = \mathrm{id}$ and $SFS = F^{-1}$ on the reversibility domain, then the stroboscopic time–phase quotient is non-orientable. If, moreover, the phase identification is $\theta \mapsto 1-\theta$, the quotient is topologically a Klein bottle.*

Sketch. $\det M < 0$ implies an orientation reversal over one period in the linearized dynamics. The existence of S satisfying (2) provides the conjugacy to backward evolution in the local domain. The two identifications together implement the cylinder gluing in n and a glide reflection in θ, generating the Klein quotient. Non-orientability follows from standard topology of the Klein surface ($w_1 \neq 0$). □

5 Local Inversion vs. Global Obstruction

We connect to the local-inverse existence from the prior UNNS temporal recursion paper.

Proposition 2 (Local F^{-1} with global obstruction)**.** *Suppose F is locally invertible on U and S satisfies (2) on U. Then for any $x \in U$ the backward orbit is locally defined. However, if the global time–phase quotient is Klein, there is no global continuous choice of time orientation along a loop homologous to the glide-reflection cycle, hence no globally consistent backward evolution that preserves a single time arrow around that loop.*

6 Two Figures

Figure 1: Forward/Reverse Recursion Cones

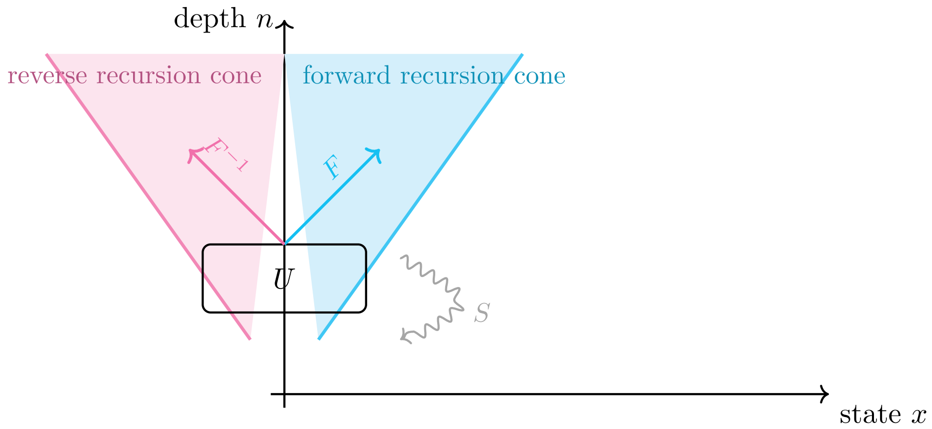
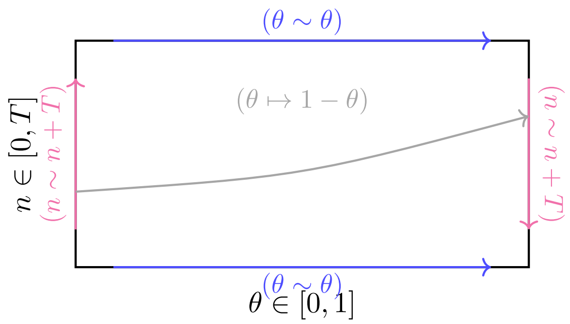


Figure 2: Klein Identification in Time–Phase



7 Examples and Diagnostics

Example A: UNNS Fibonacci-with-Flip

Consider a 2D lifted state $z_n = (x_n, x_{n-1})$ with

$$z_{n+1} = A\, z_n, \qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

and an involution $S(z) = (x_{n-1}, x_n)$ (swap coordinates) composed with a sign flip on one component. Over two steps the monodromy may become orientation reversing (depending on the sign convention), i.e. $\det(A^2) < 0$ in the signed lift. When paired with a phase θ that flips under S, the stroboscopic quotient is Klein. Locally, A is invertible, so reverse recursion exists; globally the time bundle is non-orientable.

Example B: Damped UNNS rotator with phase flip

Let F act on (r, θ) by $r \mapsto \alpha r$, $\theta \mapsto \theta + \omega \bmod 1$, with $\alpha \in (0, 1)$ and an involution $S(r, \theta) = (r, 1 - \theta)$. If the T-step angle gain is $\omega T \equiv 0 \pmod 1$, the n-direction is periodic; the S-gluing flips the phase edge, yielding a Klein quotient. Again, local back-steps exist but a global time arrow cannot be chosen consistently.

Simulation diagnostics

In discrete simulations:

- Compute the T-step Jacobian product M along a periodic (or near-periodic) orbit; check $\det M$.
- Verify a symmetry S (e.g. exchange of UNNS nests, sign/reflection in an inlaying lattice) such that $SFS = F^{-1}$ on a numerically detected reversibility domain.
- If $\det M < 0$ and the phase is observed to flip under S, expect non-orientable global behavior: loops in time–phase space return with reversed local arrow.

8 Implications

Local vs global time travel. UNNS supports *local* reverse recursion whenever F^{-1} exists on a domain, enabling stepwise backtracking. The Klein regime shows why *global* reversal can fail: non-orientability prevents a consistent time arrow around closed loops. Thus, the question "Can we travel back in time?" becomes: *locally yes, globally constrained by topology.*

Topological invariants. The obstruction is captured by the first Stiefel–Whitney class w_1 of the time–phase quotient; $w_1 \neq 0$ implies non-orientability (Klein/Möbius-type regimes). UNNS operators that implement flips (merge/collapse with sign, inlaying reflections, gauge-like involutions) can generate such regimes.

Relation to repair/normalization. UNNS repair rules that force orientation-preserving updates (e.g. forbidding sign-flip symmetries in stroboscopic closure) can *restore* orientability (cylinder/torus quotient). Conversely, adopting flip symmetries invites Klein/Möbius phases.

9 Conclusion

Temporal recursion in UNNS is naturally *local*: it relies on the existence of F^{-1} on reversibility domains. The *global* structure—determined by stroboscopic periodicity and symmetry gluing—can be non-orientable, with the Klein surface as the canonical quotient. This provides a crisp criterion for when "time reversal" is mathematically permitted locally yet globally obstructed, and it ties UNNS operator design directly to topological phases of time.

Add-on to the original paper. This note slots after the section on invertibility conditions: it identifies the precise geometric circumstance (orientation-reversing monodromy + phase flip) that yields a Klein surface time–phase bundle, explains the status of F^{-1}, and clarifies what "time travel" means in UNNS: local reversibility vs. global topological obstruction.